	U(1) gauge theory + scalar field	SU(N) gauge theory + scalar field	
DOF	$\phi(x) \in \mathbb{C}$ $\quad U_\mu(x) \in U(1)$	$\phi(x) \in \mathbb{C}^N$ $\quad U_\mu(x) \in SU(N)$	1
Gauge transform.	$\phi(x) \to e^{i\lambda(x)}$ $U_\mu(x) \to e^{i\lambda(x)}\, U_\mu(x)\, e^{i\lambda(x+ae_\mu)}$ $\lambda(x) \in \mathbb{R}$ i.e. $e^{i\lambda(x)} \in U(1)$	$\phi(x) \to \Omega(x)\,\phi(x)$ $U_\mu(x) \to \Omega(x)\, U_\mu(x)\, \Omega(x+ae_\mu)$ $\Omega(x) \in SU(N)$	
Forward covariant derivative	$D^f_\mu \phi(x) = \frac{1}{a}\left\{ U_\mu(x)\phi(x+ae_\mu) - \phi(x) \right\}$	same	2
Action $S = S_g + S_m$	$S_g = \frac{1}{2e^2} \sum_{x\in\Lambda} \sum_{\mu\nu} \mathrm{Re}\left[1 - W(\square)\right]$ (Wilson or plaquette action)	$S_g = \frac{1}{g^2} \sum_{x\in\Lambda} \sum_{\mu\nu} \mathrm{Re}\,\mathrm{tr}\left[1 - W(\square)\right]$	3
	$S_M = \sum_{x\in\Lambda} a^4 \left\{ \lvert D^f_\mu \varphi\rvert^2 + m^2 \lvert\varphi\rvert^2 + \frac{\lambda}{4!} \lvert\varphi\rvert^4 \right\}$	same	
Path-integral measure	$[d\varphi][d\varphi^*][dU] \equiv \prod_x \left\{ d\mathrm{Re}\,\varphi(x)\, d\mathrm{Im}\,\varphi(x) \prod_\mu dU_\mu(x) \right\}$		4
	$dU_\mu(x)$ is the rotational-invariant measure on the circle with unit length	$dU_\mu(x)$ is the Haar measure on $SU(N)$	

① Let $A_\mu(x)$ be a smooth gauge configuration

- $A_\mu(x)$ is an $N\times N$ hermitean traceless matrix

We identify

$$U_\mu(x) = W(x \to x+ae_\mu)$$

$$= \underset{\longrightarrow}{\mathrm{Pexp}} \left\{ i \int_{x\to x+ae_\mu} dx_\rho \, A_\rho(x) \right\}$$

where "$x \to x+ae_\mu$" is the straight segment from x to $x+ae_\mu$.

An explicit parametrization is e.g.

$$\gamma(s) = x + s e_\mu \qquad 0 \leq s \leq a$$

Under a gauge transformation $\Omega(x) \in SU(N)$:

$$A_\mu(x) \to \Omega(x) A_\mu(x) \Omega^\dagger(x) + i (\partial_\mu \Omega)(x) \Omega^\dagger(x)$$

$$U_\mu(x) \to \Omega(x) U_\mu(x) \Omega^\dagger(x+ae_\mu)$$

(2)

$$D^f_\mu \phi(x) = \frac{1}{a}\{ U_\mu(x)\phi(x+ae_\mu) - \phi(x)\}$$

Under a gauge transformation:

$$D^f_\mu \phi(x) \to \frac{1}{a}\left\{ [\Omega(x) U_\mu(x) \cancel{\Omega^\dagger(x+ae_\mu)}][\cancel{\Omega(x+ae_\mu)}\phi(x+ae_\mu)] - \Omega(x)\phi(x)\right\}$$

$$= \Omega(x) D^f_\mu \phi(x) \qquad \text{i.e. } D^f_\mu\phi(x) \text{ transforms like } \phi(x)$$

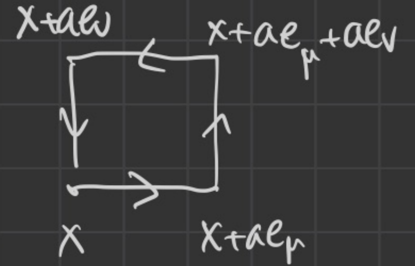
$$\lim_{a\to 0} D^f_\mu \phi(x) = \lim_{a\to 0} \frac{\overrightarrow{P\exp}\left\{ i\int_{x\to x+ae_\mu} dx_\rho A_\rho(x)\right\}\phi(x+ae_\mu) - \phi(x)}{a}$$

$$= \lim_{a\to 0} \frac{1}{a}\left\{ e^{iaA_\mu(x) + O(a^2)}[\phi(x) + a\partial_\mu\phi(x) + O(a^2)] - \phi(x)\right\}$$

$$= \lim_{a\to 0} \frac{1}{a}\left\{ [1 + iaA_\mu(x) + O(a^2)][\phi(x) + a\partial_\mu\phi(x) + O(a^2)] - \phi(x)\right\}$$

$$= \lim_{a\to 0} \frac{1}{a}\left\{ iaA_\mu(x)\phi(x) + a\partial_\mu\phi(x) + O(a^2)\right\} = D_\mu\phi(x)$$

③

$W\Big($ 

$\Big) = U_\mu(x)\, U_\nu(x+ae_\mu)\, U_\mu(x+ae_\nu)^\dagger\, U_\nu(x)^\dagger$

(a) $W(x+ae_\mu \to x) = W(x \to x+ae_\mu)^\dagger = U_\mu(x)^\dagger$

(b) $U_\mu(x) = W(x \to x+ae_\mu) = \exp\{iB_\mu(x)\}$ with $B_\mu(x)^\dagger = B_\mu(x)$ $\quad \mathrm{tr}\, B_\mu(x) = 0$

$$B_\mu(x) = a A_\mu(x) + \frac{a^2}{2} \partial_\mu A_\mu(x) + O(a^3)$$

(c) $W(\square) = \exp\{iC_{\mu\nu}(x)\}$ with $C_{\mu\nu}(x)^\dagger = C_{\mu\nu}(x)$ $\quad \mathrm{tr}\, C_{\mu\nu}(x) = 0$

$$C_{\mu\nu}(x) = a^2 F_{\mu\nu}(x) + O(a^3)$$

[Baker-Campbell-Hausdorff formula]

(d) $\mathrm{Re}\,\mathrm{tr}\, W(\square) = \mathrm{Re}\,\mathrm{tr}\, e^{iC_{\mu\nu}(x)} = \mathrm{Re}\,\mathrm{tr}\left\{1 + iC_{\mu\nu}(x) - \frac{1}{2} C^2_{\mu\nu}(x) + O(a^6)\right\}$

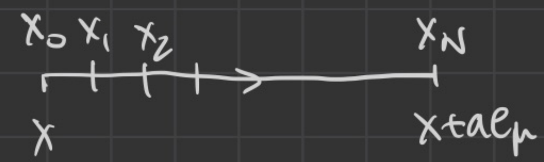
$$= N - \frac{1}{2}\mathrm{tr}\, C^2_{\mu\nu}(x) + O(a^6) = N - \frac{a^4}{2}\mathrm{tr}\, F^2_{\mu\nu}(x) + O(a^5)$$

$$\lim_{a\to 0} \sum_{x\mu\nu} \mathrm{Re}\,\mathrm{tr}\left[1 - W(\square)\right] = \lim_{a\to 0} \sum_{x\mu\nu} \frac{a^4}{2}\mathrm{tr}\, F^2_{\mu\nu}(x) = \sum_{\mu\nu} \int d^4x\, \frac{1}{2}\mathrm{tr}\, F^2_{\mu\nu}$$

[Yang-Mills action]

(a) $W(x+ae_\mu \to x) = W(x \to x+ae_\mu)^\dagger = U_\mu(x)^\dagger$

Proof :

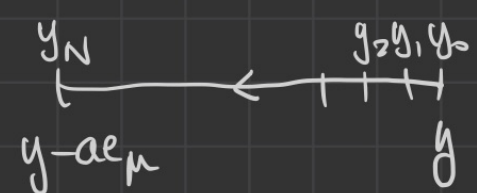


$a = \Delta s\, N$

$x_j = x + j\Delta s\, e_\mu \qquad j = 0, \ldots, N$

$$W(x \to x+ae_\mu) = \lim_{\Delta s \to 0} e^{i e_\mu \cdot A(x_0)}\, e^{i e_\mu \cdot A(x_1)} \cdots e^{i e_\mu A(x_{N-1})}$$

$$= \lim_{\Delta s \to 0} e^{iA_\mu(x_0)}\, e^{iA_\mu(x_1)} \cdots e^{iA_\mu(x_{N-1})}$$



$y = x + ae_\mu$

$y_j = y - j\Delta s\, e_\mu$

$$W(x+ae_\mu \to x) = W(x+ae_\mu \to x+ae_\mu - ae_\mu) = W(y \to y - ae_\mu) =$$

$$= \lim_{\Delta s \to 0} e^{i(-e_\mu)\cdot A(y_0)}\, e^{i(-e_\mu)A(y_1)} \cdots e^{i(-e_\mu)A(y_{N-1})}$$

$$= \lim_{\Delta s \to 0} e^{-iA_\mu(x_N)}\, e^{-iA_\mu(x_{N-1})} \cdots e^{-iA_\mu(x_1)}$$

$$= \lim_{\Delta s \to 0} \left[e^{iA_\mu(x_1)}\, e^{iA_\mu(x_2)} \cdots e^{iA_\mu(x_N)} \right]^\dagger = W(x \to x+ae_\mu)^\dagger$$

(b) $U_\mu(x) = W(x \to x + a e_\mu) = \exp\{ i B_\mu(x) \}$ with $B_\mu(x)^\dagger = B_\mu(x)$ $\quad \mathrm{tr}\, B_\mu(x) = 0$

$$B_\mu(x) = a A_\mu(x) + \frac{a^2}{2} \partial A_\mu(x) + O(a^3)$$

Proof: U is unitary hence diagonalizable, i.e. $U = W \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} W$ with $W \in U(N)$

$U \in U(N) \Rightarrow |\lambda_k| = 1 \Rightarrow \lambda_k = e^{i\varphi_k}$ with $-\pi < \varphi_k \leq \pi$

$\det U = 1 \Rightarrow 1 = \prod_k e^{i\varphi_k} = e^{i \sum_k \varphi_k} \Rightarrow \sum_k \varphi_k = 2\pi n$ with $n \in \mathbb{Z}$

$B \equiv W \begin{pmatrix} \varphi_1 & & \\ & \ddots & \\ & & \varphi_N \end{pmatrix} W^\dagger$: $\quad B^\dagger = B$, $\mathrm{tr}\, B = \sum_k \varphi_k = 2\pi n$, $e^{iB} = W \begin{pmatrix} e^{i\varphi_1} & & \\ & \ddots & \\ & & e^{i\varphi_N} \end{pmatrix} W^\dagger = U$

$\hookrightarrow$ we need to prove $n = 0$

$U(a) = 1 + O(a) \Rightarrow B = O(a)$

$U(a)$ continuous in a $\Rightarrow$ if a is small enough $iB(a) = \ln U(a)$ is continuous in a

$\Rightarrow \mathrm{tr}\, B(a)$ takes discrete values and is continuous $\Rightarrow \mathrm{tr}\, B(a) = \text{constant} = \mathrm{tr}\, B(0) = 0$

In order to prove $B_\mu(x) = a A_\mu(x) + \frac{a^2}{2} \partial_\mu A_\mu(x) + O(a^3)$

it is enough to prove that

$$U_\mu(x) = W(x \to x+ae_\mu) = \exp\{ i B_\mu(x) \}$$
$$= 1 + i B_\mu(x) - \frac{1}{2} B_\mu^2(x) + O(a^3)$$
$$= 1 + i a A_\mu(x) + \frac{i a^2}{2} \partial_\mu A_\mu(x) - \frac{a^2}{2} A_\mu^2(x) + O(a^3)$$

satisfies the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial a} W(x \to x+ae_\mu) = i W(x \to x+ae_\mu) A_\mu(x+ae_\mu) & \to \text{let's look at this} \\ W(x \to x+ae_\mu)\big|_{a=0} = 1 & \to \text{this is trivial} \end{cases}$$

$$\left.\begin{aligned} \text{lhs} &= i A_\mu(x) + i a \partial_\mu A_\mu(x) - a A_\mu^2(x) + O(a^2) \\ \text{rhs} &= i \{ 1 + i a A_\mu(x) + O(a^2) \} \{ A_\mu(x) + a \partial_\mu A_\mu(x) + O(a^2) \} \end{aligned}\right\} \text{equal}$$

(4) Haar measure. SU(N) is a Lie group. In particular it is an (N^2-1)-dimensional space. Specifying an integration measure dU means to specify a way to measure hypervolumes in SU(N).

Theorem: A unique integration measure dU exists in SU(N) with the following properties:

- dU is invariant under left and right multiplication

$$\int_{SU(N)} dU\, f(VU) = \int_{SU(N)} dU\, f(U) = \int_{SU(N)} dU\, f(UV) \qquad \text{for any } V \in SU(N)$$

[i.e. dU is invariant under change of varibles $U \to VU$ and $U \to UV$]

- $\int dU = 1$

dU is called "Haar measure".

Construction:

- Choose a parametrization $U(\xi)$ with $\xi_{\alpha=1,\dots,N^2-1}$ varying in a domain D.
- Define the metric $g_{\alpha\beta}(\xi) = \mathrm{tr}\left[\frac{\partial U^{+}}{\partial \xi^{\alpha}} \frac{\partial U}{\partial \xi^{\beta}}\right]$
- It is straightforward to prove that g is invariant under

  $U(\xi) \to U(\xi') = VU(\xi)$ and $U(\xi) \to U(\xi') = U(\xi)V$ with $V \in SU(N)$

- The Haar measure, in coordinates ξ, is given by

$$\int dU\, f(U) = \int_D \left[\prod_{\alpha} d\xi^{\alpha}\right] \sqrt{\det g(\xi)}\; f(U(\xi))$$

Exercise: Prove that any $SU(2)$ matrix can be written (uniquely) in the form

$$U = u_0 I_2 + i \underline{u}\,\underline{\sigma}$$

with $u_0 \in \mathbb{R}$, $\underline{u} \in \mathbb{R}^3$
and $u_0^2 + \underline{u}^2 = 1$

i.e. the $SU(2)$ group is identical to the 3-sphere.

Write the Haar measure for $SU(2)$ using spherical coordinates for the 3-sphere.

	U(1) gauge theory + scalar field	SU(N) gauge theory + scalar field
Gauge invariance	The action and the measure are invariant under gauge transformations	
"Naive continuum limit"	Let $A_\mu(x) \in \mathbb{R}$ and $\varphi(x) \in \mathbb{C}$ be smooth fields in the continuum. If we identify $U_\mu(x) = W(x \to x + a e_\mu) = e^{i\int_0^a ds\, A_\mu(x+se_\mu)}$ then $\lim_{a\to 0} S_{lat}(U,\varphi) = \int_x \left\{ \frac{1}{4e^2} F_{\mu\nu}^2 + \lvert D_\mu\varphi\rvert^2 + m^2\lvert\varphi^2\rvert + \frac{\lambda}{4}\lvert\varphi\rvert^4 \right\}$	Let $A_\mu(x)$ be a complex hermitean traceless $N\times N$ matrix, smooth in $x \in \mathbb{R}^4$. Let $\varphi(x) \in \mathbb{C}$ be smooth in $x \in \mathbb{R}^4$. If we identify $U_\mu(x) = W(x \to x + a e_\mu) = \overrightarrow{P\exp} \int_0^a ds\, A_\mu(x+se_\mu)$ then $\lim_{a\to 0} S_{lat}(U,\varphi) = \int_x \left\{ \frac{1}{2e^2} \mathrm{tr} F_{\mu\nu}^2 + \lvert D_\mu\varphi\rvert^2 + m^2\lvert\varphi^2\rvert + \frac{\lambda}{4}\lvert\varphi\rvert^4 \right\}$
Finiteness of Z	$Z = \int [d\varphi\, d\varphi^\dagger\, dU]\, e^{-S_{lat}(U,\varphi)}$ is finite on a finite lattice (no need of gauge-fixing)	

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \varepsilon_{ijk} \sigma_k$

$\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ is a basis for all 2×2 matrices

Important identities $\quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1} \quad [\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$

$$\sigma_i \sigma_j = \frac{1}{2}\{\sigma_i, \sigma_j\} + \frac{1}{2}[\sigma_i, \sigma_j] = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k$$

$$(\underline{a}\,\underline{\sigma})(\underline{b}\,\underline{\sigma}) = \underline{a}\,\underline{b}\,\mathbb{1} + i(\underline{a} \wedge \underline{b})\underline{\sigma}$$

$$A = a_0 \mathbb{1} + \underline{a}\,\underline{\sigma} = \begin{pmatrix} a_0 + a_3 & a_1 - i a_2 \\ a_1 + i a_2 & a_0 - a_3 \end{pmatrix}$$

$$\det A = a_0^2 - \underline{a}^2 \qquad A^{-1} = \frac{1}{a_0^2 - \underline{a}^2} \begin{pmatrix} a_0 - a_3 & -a_1 + i a_2 \\ -a_1 - i a_2 & a_0 + a_3 \end{pmatrix} = \frac{a_0 - \underline{a}\,\underline{\sigma}}{a_0^2 - \underline{a}^2}$$

$$U^\dagger U = \mathbb{1} \iff U^\dagger = U^{-1} \iff u_0^* + \underline{u}^* \underline{\sigma} = \frac{u_0 - \underline{u}\,\underline{\sigma}}{u_0^2 - \underline{u}^2} \iff u_0^* = \frac{u_0}{u_0^2 - \underline{u}^2}, \quad \underline{u}^* = -\frac{\underline{u}}{u_0 - \underline{u}^2}$$

$$U \in SU(2) \iff u_0^2 - \underline{u}^2 = 1, \quad u_0^* = u_0, \quad \underline{u}^* = -\underline{u} \iff U = \cos\theta + i\hat{n}\bar{\sigma}\sin\theta \quad \text{with } |\hat{n}| = 1, \ \hat{n} \in \mathbb{R}^3$$

$$AB = (a_0\mathbb{1} + \underline{a}\,\underline{\sigma})(b_0\mathbb{1} + \underline{b}\,\underline{\sigma}) = a_0 b_0 + \underline{a}\,\underline{b} + (a_0\underline{b} + b_0\underline{a} + i\,\underline{a}\wedge\underline{b})\underline{\sigma}$$